1.3 Evaluating Limits Analytically

- **Evaluate a limit using properties of limits.**
- **Develop and use a strategy for finding limits.**
- **Evaluate a limit using the dividing out technique.**
- **Evaluate a limit using the rationalizing technique.**
- **Evaluate a limit using the Squeeze Theorem.**

Properties of Limits

In Section 1.2, you learned that the limit of $f(x)$ as x approaches c does not depend on the value of f at $x = c$. It may happen, however, that the limit is precisely $f(c)$. In such cases, the limit can be evaluated by **direct substitution.** That is,

$$\lim_{x\to c} f(x) = f(c). \quad \text{Substitute } c \text{ for } x.$$

Such *well-behaved* functions are **continuous at *c*.** You will examine this concept more closely in Section 1.4.

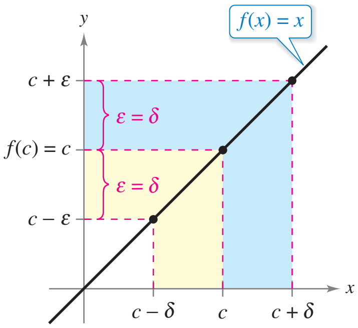


Figure 1.16

THEOREM 1.1 Some Basic Limits

Let b and c be real numbers, and let n be a positive integer.

1. $\lim_{x\to c} b = b$ **2.** $\lim_{x\to c} x = c$ **3.** $\lim_{x\to c} x^n = c^n$

Proof The proofs of Properties 1 and 3 of Theorem 1.1 are left as exercises (see Exercises 107 and 108). To prove Property 2, you need to show that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - c| < \varepsilon$ whenever $0 < |x - c| < \delta$. To do this, choose $\delta = \varepsilon$. The second inequality then implies the first, as shown in Figure 1.16.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

REMARK When encountering new notations or symbols in mathematics, be sure you know how the notations are read. For instance, the limit in Example 1(c) is read as "the limit of x^2 as x approaches 2 is 4."

EXAMPLE 1 Evaluating Basic Limits

a. $\lim_{x\to 2} 3 = 3$ **b.** $\lim_{x\to -4} x = -4$ **c.** $\lim_{x\to 2} x^2 = 2^2 = 4$

REMARK The proof of Property 1 is left as an exercise (see Exercise 109).

THEOREM 1.2 Properties of Limits

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the limits

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = K.$$

1. Scalar multiple: $\lim_{x\to c} [b f(x)] = bL$

2. Sum or difference: $\lim_{x\to c} [f(x) \pm g(x)] = L \pm K$

3. Product: $\lim_{x\to c} [f(x)g(x)] = LK$

4. Quotient: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{K}, \quad K \neq 0$

5. Power: $\lim_{x\to c} [f(x)]^n = L^n$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 2 The Limit of a Polynomial

Find the limit: $\lim_{x\to 2}(4x^2 + 3)$.

Solution

$$\begin{aligned}\lim_{x\to 2}(4x^2+3) &= \lim_{x\to 2}4x^2 + \lim_{x\to 2}3 && \text{Property 2, Theorem 1.2}\\ &= 4\left(\lim_{x\to 2}x^2\right) + \lim_{x\to 2}3 && \text{Property 1, Theorem 1.2}\\ &= 4(2^2) + 3 && \text{Properties 1 and 3, Theorem 1.1}\\ &= 19 && \text{Simplify.}\end{aligned}$$

In Example 2, note that the limit (as x approaches 2) of the *polynomial function* $p(x) = 4x^2 + 3$ is simply the value of p at $x = 2$.

$$\lim_{x\to 2} p(x) = p(2) = 4(2^2) + 3 = 19$$

This *direct substitution* property is valid for all polynomial and rational functions with nonzero denominators.

THEOREM 1.3 Limits of Polynomial and Rational Functions

If p is a polynomial function and c is a real number, then

$$\lim_{x\to c} p(x) = p(c).$$

If r is a rational function given by $r(x) = p(x)/q(x)$ and c is a real number such that $q(c) \neq 0$, then

$$\lim_{x\to c} r(x) = r(c) = \frac{p(c)}{q(c)}.$$

EXAMPLE 3 The Limit of a Rational Function

Find the limit: $\lim_{x\to 1}\frac{x^2 + x + 2}{x + 1}$.

Solution Because the denominator is not 0 when $x = 1$, you can apply Theorem 1.3 to obtain

$$\lim_{x\to 1}\frac{x^2 + x + 2}{x + 1} = \frac{1^2 + 1 + 2}{1 + 1} = \frac{4}{2} = 2.$$

Polynomial functions and rational functions are two of the three basic types of algebraic functions. The next theorem deals with the limit of the third type of algebraic function—one that involves a radical.

THEOREM 1.4 The Limit of a Function Involving a Radical

Let n be a positive integer. The limit below is valid for all c when n is odd, and is valid for $c > 0$ when n is even.

$$\lim_{x\to c}\sqrt[n]{x} = \sqrt[n]{c}$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

THE SQUARE ROOT SYMBOL

The first use of a symbol to denote the square root can be traced to the sixteenth century. Mathematicians first used the symbol $\surd$, which had only two strokes. This symbol was chosen because it resembled a lowercase *r*, to stand for the Latin word *radix*, meaning root.

The next theorem greatly expands your ability to evaluate limits because it shows how to analyze the limit of a composite function.

THEOREM 1.5 The Limit of a Composite Function

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right) = f(L).$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 4 The Limit of a Composite Function

See LarsonCalculus.com for an interactive version of this type of example.

Find the limit.

a. $\lim_{x \to 0} \sqrt{x^2 + 4}$ **b.** $\lim_{x \to 3} \sqrt[3]{2x^2 - 10}$

Solution

a. Because

$$\lim_{x \to 0} (x^2 + 4) = 0^2 + 4 = 4 \quad \text{and} \quad \lim_{x \to 4} \sqrt{x} = \sqrt{4} = 2$$

you can conclude that

$$\lim_{x \to 0} \sqrt{x^2 + 4} = \sqrt{4} = 2.$$

b. Because

$$\lim_{x \to 3} (2x^2 - 10) = 2(3^2) - 10 = 8 \quad \text{and} \quad \lim_{x \to 8} \sqrt[3]{x} = \sqrt[3]{8} = 2$$

you can conclude that

$$\lim_{x \to 3} \sqrt[3]{2x^2 - 10} = \sqrt[3]{8} = 2.$$

You have seen that the limits of many algebraic functions can be evaluated by direct substitution. The six basic trigonometric functions also exhibit this desirable quality, as shown in the next theorem (presented without proof).

THEOREM 1.6 Limits of Trigonometric Functions

Let c be a real number in the domain of the given trigonometric function.

1. $\lim_{x \to c} \sin x = \sin c$ **2.** $\lim_{x \to c} \cos x = \cos c$ **3.** $\lim_{x \to c} \tan x = \tan c$

4. $\lim_{x \to c} \cot x = \cot c$ **5.** $\lim_{x \to c} \sec x = \sec c$ **6.** $\lim_{x \to c} \csc x = \csc c$

EXAMPLE 5 Limits of Trigonometric Functions

a. $\lim_{x \to 0} \tan x = \tan(0) = 0$

b. $\lim_{x \to \pi} (x \cos x) = \left(\lim_{x \to \pi} x\right)\left(\lim_{x \to \pi} \cos x\right) = \pi \cos(\pi) = -\pi$

c. $\lim_{x \to 0} \sin^2 x = \lim_{x \to 0} (\sin x)^2 = 0^2 = 0$

A Strategy for Finding Limits

On the previous three pages, you studied several types of functions whose limits can be evaluated by direct substitution. This knowledge, together with the next theorem, can be used to develop a strategy for finding limits.

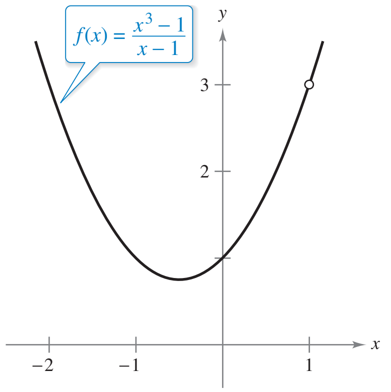
THEOREM 1.7 Functions That Agree at All but One Point

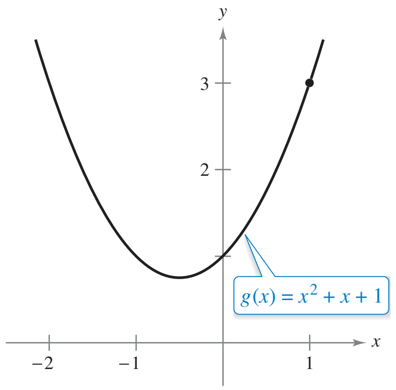
Let c be a real number, and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x).$$

A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.





f and *g* agree at all but one point.
Figure 1.17

EXAMPLE 6 Finding the Limit of a Function

Find the limit.

$$\lim_{x\to 1} \frac{x^3-1}{x-1}$$

Solution Let $f(x) = (x^3 - 1)/(x - 1)$. By factoring and dividing out like factors, you can rewrite f as

$$f(x) = \frac{(x-1)(x^2+x+1)}{(x-1)} = x^2 + x + 1 = g(x), \quad x \neq 1.$$

So, for all x-values other than $x = 1$, the functions f and g agree, as shown in Figure 1.17. Because $\lim_{x\to 1} g(x)$ exists, you can apply Theorem 1.7 to conclude that f and g have the same limit at $x = 1$.

$$\lim_{x\to 1} \frac{x^3-1}{x-1} = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1} \quad \text{Factor.}$$

$$= \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{x-1} \quad \text{Divide out like factors.}$$

$$= \lim_{x\to 1} (x^2+x+1) \quad \text{Apply Theorem 1.7.}$$

$$= 1^2 + 1 + 1 \quad \text{Use direct substitution.}$$

$$= 3 \quad \text{Simplify.}$$

REMARK When applying this strategy for finding a limit, remember that some functions do not have a limit (as x approaches c). For instance, the limit below does not exist.

$$\lim_{x\to 1} \frac{x^3+1}{x-1}$$

A Strategy for Finding Limits

1. Learn to recognize which limits can be evaluated by direct substitution. (These limits are listed in Theorems 1.1 through 1.6.)
2. When the limit of $f(x)$ as x approaches c *cannot* be evaluated by direct substitution, try to find a function g that agrees with f for all x other than $x = c$. [Choose g such that the limit of $g(x)$ *can* be evaluated by direct substitution.] Then apply Theorem 1.7 to conclude *analytically* that

$$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = g(c).$$

3. Use a *graph* or *table* to reinforce your conclusion.

Dividing Out Technique

One procedure for finding a limit analytically is the **dividing out technique.** This technique involves dividing out common factors, as shown in Example 7.

EXAMPLE 7 Dividing Out Technique

See LarsonCalculus.com for an interactive version of this type of example.

Find the limit: $\lim_{x\to -3} \frac{x^2 + x - 6}{x + 3}$.

REMARK In the solution to Example 7, be sure you see the usefulness of the Factor Theorem of Algebra. This theorem states that if c is a zero of a polynomial function, then $(x - c)$ is a factor of the polynomial. So, when you apply direct substitution to a rational function and obtain

$$r(c) = \frac{p(c)}{q(c)} = \frac{0}{0}$$

you can conclude that $(x - c)$ must be a common factor of both $p(x)$ and $q(x)$.

Solution Although you are taking the limit of a rational function, you *cannot* apply Theorem 1.3 because the limit of the denominator is 0.

$$\lim_{x\to -3} \frac{x^2 + x - 6}{x + 3} \quad \begin{array}{l} \lim_{x\to -3} (x^2 + x - 6) = 0 \\ \lim_{x\to -3} (x + 3) = 0 \end{array} \qquad \text{Direct substitution fails.}$$

Because the limit of the numerator is also 0, the numerator and denominator have a *common factor* of $(x + 3)$. So, for all $x \neq -3$, you can divide out this factor to obtain

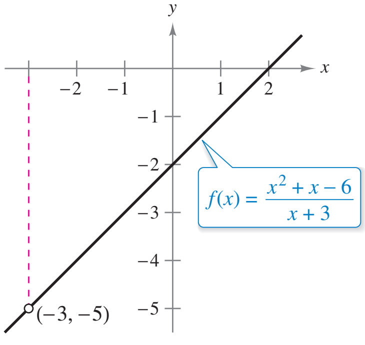
$$f(x) = \frac{x^2 + x - 6}{x + 3} = \frac{(x + 3)(x - 2)}{x + 3} = x - 2 = g(x), \quad x \neq -3.$$

Using Theorem 1.7, it follows that

$$\lim_{x\to -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x\to -3} (x - 2) \qquad \text{Apply Theorem 1.7.}$$

$$= -5. \qquad \text{Use direct substitution.}$$

This result is shown graphically in Figure 1.18. Note that the graph of the function f coincides with the graph of the function $g(x) = x - 2$, except that the graph of f has a gap at the point $(-3, -5)$.



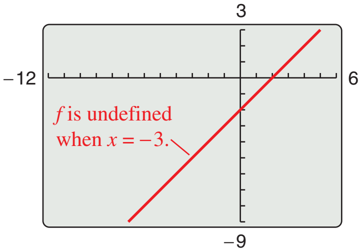
f is undefined when $x = -3$.
Figure 1.18

In Example 7, direct substitution produced the meaningless fractional form 0/0. An expression such as 0/0 is called an **indeterminate form** because you cannot (from the form alone) determine the limit. When you try to evaluate a limit and encounter this form, remember that you must rewrite the fraction so that the new denominator does not have 0 as its limit. One way to do this is to *divide out like factors*. Another way is to use the *rationalizing technique* shown on the next page.

TECHNOLOGY PITFALL A graphing utility can give misleading information about the graph of a function. For instance, try graphing the function from Example 7

$$f(x) = \frac{x^2 + x - 6}{x + 3}$$

on a standard viewing window (see Figure 1.19). On most graphing utilities, the graph appears to be defined at every real number. However, because f is undefined when $x = -3$, you know that the graph of f has a hole at $x = -3$. You can verify this on a graphing utility using the *trace* or *table* feature.



Misleading graph of f
Figure 1.19

Rationalizing Technique

Another way to find a limit analytically is the **rationalizing technique,** which involves rationalizing the numerator of a fractional expression. Recall that rationalizing the numerator means multiplying the numerator and denominator by the conjugate of the numerator. For instance, to rationalize the numerator of

$$\frac{\sqrt{x} + 4}{x}$$

multiply the numerator and denominator by the conjugate of $\sqrt{x} + 4$, which is

$$\sqrt{x} - 4.$$

EXAMPLE 8 Rationalizing Technique

Find the limit: $\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$.

Solution By direct substitution, you obtain the indeterminate form 0/0.

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} \quad \begin{cases} \lim_{x \to 0} \left(\sqrt{x+1} - 1\right) = 0 \\ \lim_{x \to 0} x = 0 \end{cases}$$

Direct substitution fails.

In this case, you can rewrite the fraction by rationalizing the numerator.

$$\begin{aligned} \frac{\sqrt{x+1} - 1}{x} &= \left(\frac{\sqrt{x+1} - 1}{x}\right)\left(\frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}\right) \\ &= \frac{(x+1) - 1}{x\left(\sqrt{x+1} + 1\right)} \\ &= \frac{\cancel{x}}{\cancel{x}\left(\sqrt{x+1} + 1\right)} \\ &= \frac{1}{\sqrt{x+1} + 1}, \quad x \neq 0 \end{aligned}$$

Now, using Theorem 1.7, you can evaluate the limit as shown.

$$\begin{aligned} \lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} &= \lim_{x \to 0} \frac{1}{\sqrt{x+1} + 1} \\ &= \frac{1}{1 + 1} \\ &= \frac{1}{2} \end{aligned}$$

A table or a graph can reinforce your conclusion that the limit is $\frac{1}{2}$. (See Figure 1.20.)

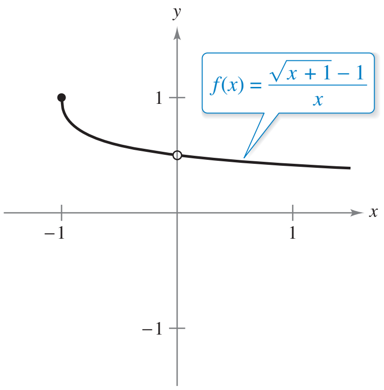
x approaches 0 from the left. → ← *x* approaches 0 from the right.

x	-0.25	-0.1	-0.01	-0.001	0	0.001	0.01	0.1	0.25
$f(x)$	0.5359	0.5132	0.5013	0.5001	?	0.4999	0.4988	0.4881	0.4721

$f(x)$ approaches 0.5. → ← $f(x)$ approaches 0.5.

REMARK The rationalizing technique for evaluating limits is based on multiplication by a convenient form of 1. In Example 8, the convenient form is

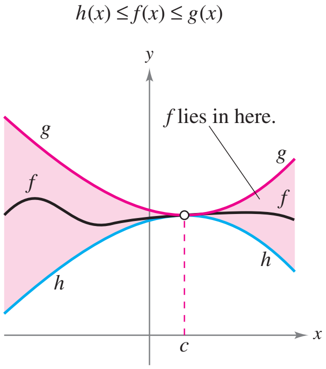
$$1 = \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1}.$$



The limit of $f(x)$ as x approaches 0 is $\frac{1}{2}$.
Figure 1.20

The Squeeze Theorem

The next theorem concerns the limit of a function that is squeezed between two other functions, each of which has the same limit at a given x-value, as shown in Figure 1.21.



The Squeeze Theorem
Figure 1.21

THEOREM 1.8 The Squeeze Theorem

If $h(x) \le f(x) \le g(x)$ for all x in an open interval containing c, except possibly at c itself, and if

$$\lim_{x\to c} h(x) = L = \lim_{x\to c} g(x)$$

then $\lim_{x\to c} f(x)$ exists and is equal to L.

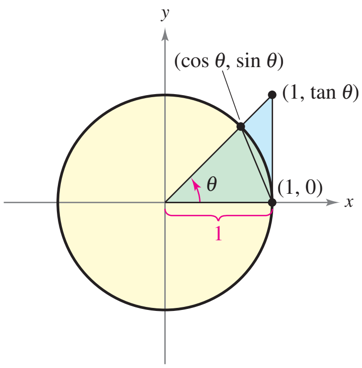
A proof of this theorem is given in Appendix A.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

You can see the usefulness of the Squeeze Theorem (also called the Sandwich Theorem or the Pinching Theorem) in the proof of Theorem 1.9.

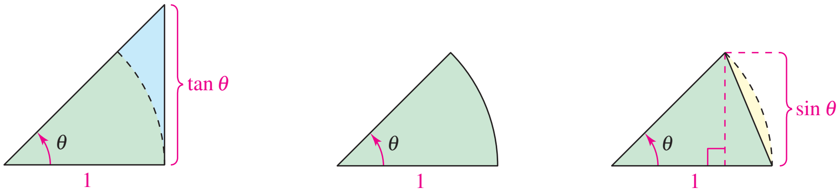
THEOREM 1.9 Two Special Trigonometric Limits

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$ **2.** $\lim_{x\to 0} \frac{1 - \cos x}{x} = 0$



A circular sector is used to prove Theorem 1.9.
Figure 1.22

Proof The proof of the second limit is left as an exercise (see Exercise 121). To avoid the confusion of two different uses of x, the proof of the first limit is presented using the variable θ, where θ is an acute positive angle *measured in radians*. Figure 1.22 shows a circular sector that is squeezed between two triangles.



Area of triangle	$\ge$	Area of sector	$\ge$	Area of triangle
$\frac{\tan\theta}{2}$	$\ge$	$\frac{\theta}{2}$	$\ge$	$\frac{\sin\theta}{2}$

Multiplying each expression by $2/\sin\theta$ produces

$$\frac{1}{\cos\theta} \ge \frac{\theta}{\sin\theta} \ge 1$$

and taking reciprocals and reversing the inequalities yields

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1.$$

Because $\cos\theta = \cos(-\theta)$ and $(\sin\theta)/\theta = [\sin(-\theta)]/(-\theta)$, you can conclude that this inequality is valid for *all* nonzero θ in the open interval $(-\pi/2, \pi/2)$. Finally, because $\lim_{\theta\to 0} \cos\theta = 1$ and $\lim_{\theta\to 0} 1 = 1$, you can apply the Squeeze Theorem to conclude that $\lim_{\theta\to 0} (\sin\theta)/\theta = 1$. *See LarsonCalculus.com for Bruce Edwards's video of this proof.* ■

EXAMPLE 9 A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x\to 0} \frac{\tan x}{x}$.

Solution Direct substitution yields the indeterminate form 0/0. To solve this problem, you can write $\tan x$ as $(\sin x)/(\cos x)$ and obtain

$$\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right).$$

Now, because

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

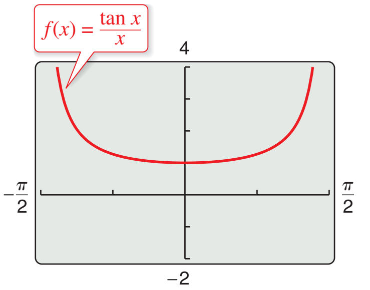
and

$$\lim_{x\to 0} \frac{1}{\cos x} = 1$$

you can obtain

$$\begin{aligned}\lim_{x\to 0} \frac{\tan x}{x} &= \left(\lim_{x\to 0} \frac{\sin x}{x}\right)\left(\lim_{x\to 0} \frac{1}{\cos x}\right)\\ &= (1)(1)\\ &= 1.\end{aligned}$$

(See Figure 1.23.)



The limit of $f(x)$ as x approaches 0 is 1.
Figure 1.23

REMARK Be sure you understand the mathematical conventions regarding parentheses and trigonometric functions. For instance, in Example 10, $\sin 4x$ means $\sin(4x)$.

EXAMPLE 10 A Limit Involving a Trigonometric Function

Find the limit: $\lim_{x\to 0} \frac{\sin 4x}{x}$.

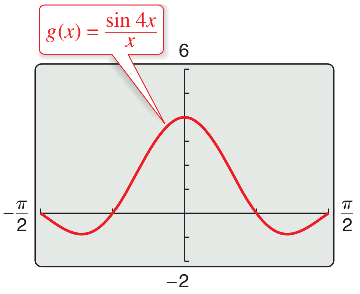
Solution Direct substitution yields the indeterminate form 0/0. To solve this problem, you can rewrite the limit as

$$\lim_{x\to 0} \frac{\sin 4x}{x} = 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right).$$ Multiply and divide by 4.

Now, by letting $y = 4x$ and observing that x approaches 0 if and only if y approaches 0, you can write

$$\begin{aligned}\lim_{x\to 0} \frac{\sin 4x}{x} &= 4\left(\lim_{x\to 0} \frac{\sin 4x}{4x}\right)\\ &= 4\left(\lim_{y\to 0} \frac{\sin y}{y}\right) && \text{Let } y = 4x.\\ &= 4(1) && \text{Apply Theorem 1.9(1).}\\ &= 4.\end{aligned}$$

(See Figure 1.24.)



The limit of $g(x)$ as x approaches 0 is 4.
Figure 1.24

TECHNOLOGY Use a graphing utility to confirm the limits in the examples and in the exercise set. For instance, Figures 1.23 and 1.24 show the graphs of

$$f(x) = \frac{\tan x}{x} \quad \text{and} \quad g(x) = \frac{\sin 4x}{x}.$$

Note that the first graph appears to contain the point (0, 1) and the second graph appears to contain the point (0, 4), which lends support to the conclusions obtained in Examples 9 and 10.

1.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Estimating Limits In Exercises 1–4, use a graphing utility to graph the function and visually estimate the limits.

1. $h(x) = -x^2 + 4x$

(a) $\lim_{x \to 4} h(x)$

(b) $\lim_{x \to -1} h(x)$

2. $g(x) = \frac{12(\sqrt{x} - 3)}{x - 9}$

(a) $\lim_{x \to 4} g(x)$

(b) $\lim_{x \to 9} g(x)$

3. $f(x) = x \cos x$

(a) $\lim_{x \to 0} f(x)$

(b) $\lim_{x \to \pi/3} f(x)$

4. $f(t) = t|t - 4|$

(a) $\lim_{t \to 4} f(t)$

(b) $\lim_{t \to -1} f(t)$

Finding a Limit In Exercises 5–22, find the limit.

5. $\lim_{x \to 2} x^3$

6. $\lim_{x \to -3} x^4$

7. $\lim_{x \to 0} (2x - 1)$

8. $\lim_{x \to -4} (2x + 3)$

9. $\lim_{x \to -3} (x^2 + 3x)$

10. $\lim_{x \to 2} (-x^3 + 1)$

11. $\lim_{x \to -3} (2x^2 + 4x + 1)$

12. $\lim_{x \to 1} (2x^3 - 6x + 5)$

13. $\lim_{x \to 3} \sqrt{x + 1}$

14. $\lim_{x \to 2} \sqrt[3]{12x + 3}$

15. $\lim_{x \to -4} (x + 3)^2$

16. $\lim_{x \to 0} (3x - 2)^4$

17. $\lim_{x \to 2} \frac{1}{x}$

18. $\lim_{x \to -5} \frac{5}{x + 3}$

19. $\lim_{x \to 1} \frac{x}{x^2 + 4}$

20. $\lim_{x \to 1} \frac{3x + 5}{x + 1}$

21. $\lim_{x \to 7} \frac{3x}{\sqrt{x + 2}}$

22. $\lim_{x \to 3} \frac{\sqrt{x + 6}}{x + 2}$

Finding Limits In Exercises 23–26, find the limits.

23. $f(x) = 5 - x, \; g(x) = x^3$

(a) $\lim_{x \to 1} f(x)$ (b) $\lim_{x \to 4} g(x)$ (c) $\lim_{x \to 1} g(f(x))$

24. $f(x) = x + 7, \; g(x) = x^2$

(a) $\lim_{x \to -3} f(x)$ (b) $\lim_{x \to 4} g(x)$ (c) $\lim_{x \to -3} g(f(x))$

25. $f(x) = 4 - x^2, \; g(x) = \sqrt{x + 1}$

(a) $\lim_{x \to 1} f(x)$ (b) $\lim_{x \to 3} g(x)$ (c) $\lim_{x \to 1} g(f(x))$

26. $f(x) = 2x^2 - 3x + 1, \; g(x) = \sqrt[3]{x + 6}$

(a) $\lim_{x \to 4} f(x)$ (b) $\lim_{x \to 21} g(x)$ (c) $\lim_{x \to 4} g(f(x))$

Finding a Limit of a Trigonometric Function In Exercises 27–36, find the limit of the trigonometric function.

27. $\lim_{x \to \pi/2} \sin x$

28. $\lim_{x \to \pi} \tan x$

29. $\lim_{x \to 1} \cos \frac{\pi x}{3}$

30. $\lim_{x \to 2} \sin \frac{\pi x}{2}$

31. $\lim_{x \to 0} \sec 2x$

32. $\lim_{x \to \pi} \cos 3x$

33. $\lim_{x \to 5\pi/6} \sin x$

34. $\lim_{x \to 5\pi/3} \cos x$

35. $\lim_{x \to 3} \tan\left(\frac{\pi x}{4}\right)$

36. $\lim_{x \to 7} \sec\left(\frac{\pi x}{6}\right)$

Evaluating Limits In Exercises 37–40, use the information to evaluate the limits.

37. $\lim_{x \to c} f(x) = 3$

$\lim_{x \to c} g(x) = 2$

(a) $\lim_{x \to c} [5g(x)]$

(b) $\lim_{x \to c} [f(x) + g(x)]$

(c) $\lim_{x \to c} [f(x)g(x)]$

(d) $\lim_{x \to c} \frac{f(x)}{g(x)}$

38. $\lim_{x \to c} f(x) = 2$

$\lim_{x \to c} g(x) = \frac{3}{4}$

(a) $\lim_{x \to c} [4f(x)]$

(b) $\lim_{x \to c} [f(x) + g(x)]$

(c) $\lim_{x \to c} [f(x)g(x)]$

(d) $\lim_{x \to c} \frac{f(x)}{g(x)}$

39. $\lim_{x \to c} f(x) = 4$

(a) $\lim_{x \to c} [f(x)]^3$

(b) $\lim_{x \to c} \sqrt{f(x)}$

(c) $\lim_{x \to c} [3f(x)]$

(d) $\lim_{x \to c} [f(x)]^{3/2}$

40. $\lim_{x \to c} f(x) = 27$

(a) $\lim_{x \to c} \sqrt[3]{f(x)}$

(b) $\lim_{x \to c} \frac{f(x)}{18}$

(c) $\lim_{x \to c} [f(x)]^2$

(d) $\lim_{x \to c} [f(x)]^{2/3}$

Finding a Limit In Exercises 41–46, write a simpler function that agrees with the given function at all but one point. Then find the limit of the function. Use a graphing utility to confirm your result.

41. $\lim_{x \to 0} \frac{x^2 + 3x}{x}$

42. $\lim_{x \to 0} \frac{x^4 - 5x^2}{x^2}$

43. $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$

44. $\lim_{x \to -2} \frac{3x^2 + 5x - 2}{x + 2}$

45. $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

46. $\lim_{x \to -1} \frac{x^3 + 1}{x + 1}$

Finding a Limit In Exercises 47–62, find the limit.

47. $\lim_{x \to 0} \frac{x}{x^2 - x}$

48. $\lim_{x \to 0} \frac{2x}{x^2 + 4x}$

49. $\lim_{x \to 4} \frac{x - 4}{x^2 - 16}$

50. $\lim_{x \to 5} \frac{5 - x}{x^2 - 25}$

51. $\lim_{x \to -3} \frac{x^2 + x - 6}{x^2 - 9}$

52. $\lim_{x \to 2} \frac{x^2 + 2x - 8}{x^2 - x - 2}$

53. $\lim_{x \to 4} \frac{\sqrt{x + 5} - 3}{x - 4}$

54. $\lim_{x \to 3} \frac{\sqrt{x + 1} - 2}{x - 3}$

55. $\lim_{x \to 0} \frac{\sqrt{x + 5} - \sqrt{5}}{x}$

56. $\lim_{x \to 0} \frac{\sqrt{2 + x} - \sqrt{2}}{x}$

57. $\lim_{x \to 0} \frac{[1/(3 + x)] - (1/3)}{x}$

58. $\lim_{x \to 0} \frac{[1/(x + 4)] - (1/4)}{x}$

59. $\lim_{\Delta x\to 0} \dfrac{2(x+\Delta x)-2x}{\Delta x}$

60. $\lim_{\Delta x\to 0} \dfrac{(x+\Delta x)^2-x^2}{\Delta x}$

61. $\lim_{\Delta x\to 0} \dfrac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x}$

62. $\lim_{\Delta x\to 0} \dfrac{(x+\Delta x)^3-x^3}{\Delta x}$

Finding a Limit of a Trigonometric Function **In Exercises 63–74, find the limit of the trigonometric function.**

63. $\lim_{x\to 0} \dfrac{\sin x}{5x}$

64. $\lim_{x\to 0} \dfrac{3(1-\cos x)}{x}$

65. $\lim_{x\to 0} \dfrac{\sin x(1-\cos x)}{x^2}$

66. $\lim_{\theta\to 0} \dfrac{\cos\theta\tan\theta}{\theta}$

67. $\lim_{x\to 0} \dfrac{\sin^2 x}{x}$

68. $\lim_{x\to 0} \dfrac{\tan^2 x}{x}$

69. $\lim_{h\to 0} \dfrac{(1-\cos h)^2}{h}$

70. $\lim_{\phi\to\pi} \phi\sec\phi$

71. $\lim_{x\to\pi/2} \dfrac{\cos x}{\cot x}$

72. $\lim_{x\to\pi/4} \dfrac{1-\tan x}{\sin x-\cos x}$

73. $\lim_{t\to 0} \dfrac{\sin 3t}{2t}$

74. $\lim_{x\to 0} \dfrac{\sin 2x}{\sin 3x}$ $\left[\textit{Hint: } \text{Find } \lim_{x\to 0}\left(\dfrac{2\sin 2x}{2x}\right)\left(\dfrac{3x}{3\sin 3x}\right).\right]$

Graphical, Numerical, and Analytic Analysis **In Exercises 75–82, use a graphing utility to graph the function and estimate the limit. Use a table to reinforce your conclusion. Then find the limit by analytic methods.**

75. $\lim_{x\to 0} \dfrac{\sqrt{x+2}-\sqrt{2}}{x}$

76. $\lim_{x\to 16} \dfrac{4-\sqrt{x}}{x-16}$

77. $\lim_{x\to 0} \dfrac{[1/(2+x)]-(1/2)}{x}$

78. $\lim_{x\to 2} \dfrac{x^5-32}{x-2}$

79. $\lim_{t\to 0} \dfrac{\sin 3t}{t}$

80. $\lim_{x\to 0} \dfrac{\cos x-1}{2x^2}$

81. $\lim_{x\to 0} \dfrac{\sin x^2}{x}$

82. $\lim_{x\to 0} \dfrac{\sin x}{\sqrt[3]{x}}$

Finding a Limit **In Exercises 83–88, find**

$$\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

83. $f(x)=3x-2$

84. $f(x)=-6x+3$

85. $f(x)=x^2-4x$

86. $f(x)=\sqrt{x}$

87. $f(x)=\dfrac{1}{x+3}$

88. $f(x)=\dfrac{1}{x^2}$

Using the Squeeze Theorem **In Exercises 89 and 90, use the Squeeze Theorem to find $\lim_{x\to c} f(x)$.**

89. $c=0$

$4-x^2\le f(x)\le 4+x^2$

90. $c=a$

$b-|x-a|\le f(x)\le b+|x-a|$

Using the Squeeze Theorem **In Exercises 91–94, use a graphing utility to graph the given function and the equations $y=|x|$ and $y=-|x|$ in the same viewing window. Using the graphs to observe the Squeeze Theorem visually, find $\lim_{x\to 0} f(x)$.**

91. $f(x)=|x|\sin x$

92. $f(x)=|x|\cos x$

93. $f(x)=x\sin\dfrac{1}{x}$

94. $h(x)=x\cos\dfrac{1}{x}$

WRITING ABOUT CONCEPTS

95. Functions That Agree at All but One Point

(a) In the context of finding limits, discuss what is meant by two functions that agree at all but one point.

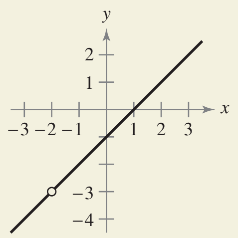
(b) Give an example of two functions that agree at all but one point.

96. Indeterminate Form What is meant by an indeterminate form?

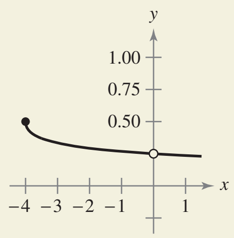
97. Squeeze Theorem In your own words, explain the Squeeze Theorem.

98. HOW DO YOU SEE IT? Would you use the dividing out technique or the rationalizing technique to find the limit of the function? Explain your reasoning.

(a) $\lim_{x\to -2} \dfrac{x^2+x-2}{x+2}$



(b) $\lim_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$



99. Writing Use a graphing utility to graph

$$f(x)=x,\quad g(x)=\sin x,\quad \text{and}\quad h(x)=\frac{\sin x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is close to 0. Use the comparison to write a short paragraph explaining why

$$\lim_{x\to 0} h(x)=1.$$

100. Writing Use a graphing utility to graph

$$f(x)=x,\quad g(x)=\sin^2 x,\quad \text{and}\quad h(x)=\frac{\sin^2 x}{x}$$

in the same viewing window. Compare the magnitudes of $f(x)$ and $g(x)$ when x is close to 0. Use the comparison to write a short paragraph explaining why

$$\lim_{x\to 0} h(x)=0.$$

Free-Falling Object

In Exercises 101 and 102, use the position function $s(t) = -16t^2 + 500$, which gives the height (in feet) of an object that has fallen for t seconds from a height of 500 feet. The velocity at time $t = a$ seconds is given by

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

101. A construction worker drops a full paint can from a height of 500 feet. How fast will the paint can be falling after 2 seconds?

102. A construction worker drops a full paint can from a height of 500 feet. When will the paint can hit the ground? At what velocity will the paint can impact the ground?

Free-Falling Object **In Exercises 103 and 104, use the position function $s(t) = -4.9t^2 + 200$, which gives the height (in meters) of an object that has fallen for t seconds from a height of 200 meters. The velocity at time $t = a$ seconds is given by**

$$\lim_{t \to a} \frac{s(a) - s(t)}{a - t}.$$

103. Find the velocity of the object when $t = 3$.

104. At what velocity will the object impact the ground?

105. Finding Functions Find two functions f and g such that $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist, but

$$\lim_{x \to 0} [f(x) + g(x)]$$

does exist.

106. Proof Prove that if $\lim_{x \to c} f(x)$ exists and $\lim_{x \to c} [f(x) + g(x)]$ does not exist, then $\lim_{x \to c} g(x)$ does not exist.

107. Proof Prove Property 1 of Theorem 1.1.

108. Proof Prove Property 3 of Theorem 1.1. (You may use Property 3 of Theorem 1.2.)

109. Proof Prove Property 1 of Theorem 1.2.

110. Proof Prove that if $\lim_{x \to c} f(x) = 0$, then $\lim_{x \to c} |f(x)| = 0$.

111. Proof Prove that if $\lim_{x \to c} f(x) = 0$ and $|g(x)| \le M$ for a fixed number M and all $x \ne c$, then $\lim_{x \to c} f(x)g(x) = 0$.

112. Proof

(a) Prove that if $\lim_{x \to c} |f(x)| = 0$, then $\lim_{x \to c} f(x) = 0$.

(*Note:* This is the converse of Exercise 110.)

(b) Prove that if $\lim_{x \to c} f(x) = L$, then $\lim_{x \to c} |f(x)| = |L|$.

[*Hint:* Use the inequality $\big||f(x)| - |L|\big| \le |f(x) - L|$.]

113. Think About It Find a function f to show that the converse of Exercise 112(b) is not true. [*Hint:* Find a function f such that $\lim_{x \to c} |f(x)| = |L|$ but $\lim_{x \to c} f(x)$ does not exist.]

114. Think About It When using a graphing utility to generate a table to approximate

$$\lim_{x \to 0} \frac{\sin x}{x}$$

a student concluded that the limit was 0.01745 rather than 1. Determine the probable cause of the error.

True or False? **In Exercises 115–120, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

115. $\lim_{x \to 0} \frac{|x|}{x} = 1$

116. $\lim_{x \to \pi} \frac{\sin x}{x} = 1$

117. If $f(x) = g(x)$ for all real numbers other than $x = 0$, and $\lim_{x \to 0} f(x) = L$, then $\lim_{x \to 0} g(x) = L$.

118. If $\lim_{x \to c} f(x) = L$, then $f(c) = L$.

119. $\lim_{x \to 2} f(x) = 3$, where $f(x) = \begin{cases} 3, & x \le 2 \\ 0, & x > 2 \end{cases}$

120. If $f(x) < g(x)$ for all $x \ne a$, then $\lim_{x \to a} f(x) < \lim_{x \to a} g(x)$.

121. Proof Prove the second part of Theorem 1.9.

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

122. Piecewise Functions Let

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

and

$$g(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}.$$

Find (if possible) $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$.

123. Graphical Reasoning Consider $f(x) = \dfrac{\sec x - 1}{x^2}$.

(a) Find the domain of f.

(b) Use a graphing utility to graph f. Is the domain of f obvious from the graph? If not, explain.

(c) Use the graph of f to approximate $\lim_{x \to 0} f(x)$.

(d) Confirm your answer to part (c) analytically.

124. Approximation

(a) Find $\lim_{x \to 0} \dfrac{1 - \cos x}{x^2}$.

(b) Use your answer to part (a) to derive the approximation $\cos x \approx 1 - \frac{1}{2}x^2$ for x near 0.

(c) Use your answer to part (b) to approximate $\cos(0.1)$.

(d) Use a calculator to approximate $\cos(0.1)$ to four decimal places. Compare the result with part (c).

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